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Two-component soliton systems and the Painlevé equations

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Abstract
We give an extension of the two-component KP hierarchy by considering additional time variables. We obtain the linear 2×2 system by taking into consideration the hierarchy through a reduction procedure. The Lax pair of the Schlesinger system and the sixth Painlevé equation is given from this linear system. A unified approach to treat the other Painlevé equations from the usual two-component KP hierarchy is also considered.

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1. Introduction

The aim of this paper is to establish correspondence between the isospectral deformation and the monodromy preserving deformation. We construct an extension of the two-component Kadomtsev–Petviashvili (KP) hierarchy by introducing new time variables. We give the relation between this hierarchy and the sixth Painlevé equation. We show also the relation between the usual two-component KP hierarchy and the other Painlevé equations.

The relation between the isomonodromic deformation and the isospectral one was discussed; see [2, 5, 18, 19]. Jimbo and Miwa [5] described a procedure to reduce the isospectral deformation into the isomonodromic deformation consistently by using the τ-function. One can obtain not only the Painlevé equations themselves but also the Lax pairs of them. The third Painlevé equation (P_{III}) and the fourth Painlevé equation (P_{IV}) were obtained through the reduction from the Pohlmeyer–Lund–Regge equation and the nonlinear Schrödinger equation, respectively. Takasaki's paper [17] has been related to a Hamiltonian structure of the first Painlevé hierarchy in terms of the Sato theory. Noumi and Yamada [14] introduced a Painlevé system associated with the affine root system of type $A^{(1)}_{n-1}$ including

$\mathrm{P_{II}}\big(A_1^{(1)}\big)$, $\mathrm{P_{IV}}\big(A_2^{(1)}\big)$ and $\mathrm{P_V}\big(A_3^{(1)}\big)$. The systems are equivalent to similarity reductions of the n-reduced modified KP hierarchy. The coefficients of the Lax pair for the system of type $A_{n-1}^{(1)}$ are $n \times n$ matrices [13]. The similarity reductions of the Drinfel'd–Sokolov hierarchies were investigated by Ikeda, Kakei and Kikuchi; see [6–8, 10]. As a consequence, $\mathrm{P_V}$ can be obtained from the modified Yajima–Oikawa equation, and $\mathrm{P_{VI}}$ with four parameters can be derived from the three-wave resonant system. In the paper [8], the coefficients of the Lax pair which they obtained were also 3×3 matrices. They showed that the 2×2 linear system can be obtained from the 3×3 linear system by Laplace transformation [3, 12]. Zhang's paper [20] has reviewed the multi-component KP hierarchy and one of its reduction.

We give systems of the isospectral deformations that are directly reduced to the Lax pairs for the Painlevé equations. Specifically, we deal with the linear systems with 2×2 matrices, in fact the types of singular points of the linear system with 2×2 matrices which correspond to the types of the Painlevé equations. We intend to study the Painlevé equations by relating the properties of the soliton equations to that of the Painlevé equations. In order to construct the signpost of this approach, we try to formulate the holonomic deformation by using the Sato theory.

In this paper, we consider an infinite-dimensional integrable hierarchy and give the Lax pair with 2×2 matrices for $\mathrm{P_{VI}}$. This hierarchy is an extension of the two-component KP hierarchy by using additional time variables. The extension means that the hierarchy restricted to be independent of the introduced time variables is equal to the usual two-component KP hierarchy. We consider especially the $(1, 1)$-reduction of the two-component KP hierarchy which is known as the nonlinear Schrödinger hierarchy. It is contained in the extended Zakharov–Shabat hierarchy; cf [1]. We formulate the extended hierarchy by using the Sato–Wilson formalism and then define a wavefunction which is a normal solution of the linear system.

Then we consider the holonomic deformation in the same way as the isospectral deformation. We construct a system of linear differential equations in the spectral parameter by using the wavefunction in the extended hierarchy. We obtain nonlinear systems that describe the condition of the complete integrability of the linear systems. The infinite-dimensional system is reduced to the Schlesinger system, from which $\mathrm{P_{VI}}$ is obtained.

We treat also the other Painlevé equations from the viewpoint of the usual two-component KP hierarchy. We study the nonlinear Schrödinger hierarchy by using the Sato–Wilson formalism, and then give different wavefunctions. The choice of the wavefunction can be done freely from the two-component KP hierarchy, the holonomic deformations might be dependent on it. We construct systems of linear differential equations in the spectral parameter by using each wavefunction. We then obtain nonlinear systems that describe the condition of the complete integrability of the linear systems. If we assume several reductions for the linear systems, then the infinite-dimensional systems are reduced to one-dimensional systems which yield the other Painlevé equations; see section 4 below. It follows that the reductions of the nonlinear Schrödinger equation give rise to not only $\mathrm{P_{IV}}$ (see [5]) but also $\mathrm{P_V}$ and $\mathrm{P_{III}}$.

In section 2, we construct an extension of the two-component KP hierarchy by employing the Sato–Wilson formalism. In section 3, we consider the holonomic deformation based on this extended hierarchy and obtain the nonlinear system that describes the condition of this deformation. We see that the nonlinear system reduces to $\mathrm{P_{VI}}$. In section 4, we study the holonomic deformation that contains the two-component KP hierarchy and show that the nonlinear systems that describe the condition of this deformation reduce to the other Painlevé equations, $\mathrm{P_V}$, $\mathrm{P_{IV}}$, $\mathrm{P_{III}}$ and $\mathrm{P_{II}}$.

2. An extension of the two-component KP hierarchy

In the present section, we study an extension of the (1, 1)-reduction of the two-component KP hierarchy. We give a formulation of this hierarchy by using the Sato–Wilson formalism, and then obtain an integrable system by means of the Zakharov–Shabat system.

2.1. Pseudo-differential operator

The multi-component theory of the KP hierarchy is established in the paper, [15]. The n-component KP hierarchy is formulated by matrix pseudo-differential operators of size $n \times n$, instead of scalar ones used in the one-component hierarchy. We explain some notation about the matrix pseudo-differential operators of size $n \times n$.

The action of the differential operator ∂_x on an $n \times n$ matrix $f(x)$ is

$$\partial_x f(x) = \frac{\mathrm{d}}{\mathrm{d}x} f(x).$$

The operator ∂_x^{-1} is defined by

$$\partial_x \partial_x^{-1} = \partial_x^{-1} \partial_x \equiv 1.$$

Pseudo-differential operators are defined by using the operators ∂_x and ∂_x^{-1}.

Definition 1. *A pseudo-differential operator with matrix coefficients of size $n \times n$ is a linear operator,*

$$\mathcal{A} = \sum_m a_m(x) \partial_x^m,$$

where $a_m(x)$ is an $n \times n$ matrix-valued function of x.

A sum of pseudo-differential operators is defined in the usual way by collecting terms, and their product is defined by the following extension of Leibniz's rule,

$$\mathcal{A}\mathcal{B} = \sum_{m,n} a_m(x) \partial_x^m b_n(x) \partial_x^n = \sum_{m,n} \sum_{k=0}^{\infty} \binom{m}{k} a_m(x) b_n^{(k)}(x) \partial_x^{m+n-k},$$

where

$$\binom{m}{k} = \begin{cases} \dfrac{m(m-1)\ldots(m-k+1)}{k!} & (k \geqslant 1) \\ 1 & (k = 0). \end{cases}$$

We define the differential operator part of a pseudo-differential operator $\mathcal{A}$ by

$$(\mathcal{A})_+ = \sum_{m \geqslant 0} a_m(x) \partial_x^m.$$

A pseudo-differential operator possesses a unique inverse, denoted by $\mathcal{A}^{-1}$.

2.2. Sato equation

In the Sato–Wilson formalism, a pseudo-differential operator called the gauge operator plays an essential role. The coefficients of the gauge operator are dependent variables in the soliton system. The condition of the isospectral deformation is given by the Sato equations that the gauge operator should satisfy.

We define the gauge operator of size 2×2 by

$$\mathcal{W} = I + \sum_{k=1}^{\infty} w_k \partial_x^{-k}, \tag{1}$$

whose 2×2 coefficient matrices $w_k (k \geqslant 1)$ do not depend on the parameter x. This condition for the coefficients is equivalent to 'the (1, 1)-reduction'. The formal series $\mathcal{W}$ can be inverted. Let

$$\mathcal{W}^{-1} = \sum_{k=0}^{\infty} v_k \partial_x^{-k}, \tag{2}$$

the first few v_k's are

$$\begin{aligned} v_0 &= I, \\ v_1 &= -w_1, \\ v_2 &= -w_2 + w_1^2, \\ v_3 &= -w_3 + w_1 w_2 + w_2 w_1 - w_1^3. \end{aligned} \tag{3}$$

The gauge operator $\mathcal{W}$ can be used to define the operator

$$\mathcal{U} = \mathcal{W} \sigma_3 \mathcal{W}^{-1} = \sigma_3 + \sum_{k=1}^{\infty} u_k \partial_x^{-k}, \tag{4}$$

where

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$u_k = \sum_{j=1}^{k} [w_j, \sigma_3] v_{k-j} \qquad (k \geqslant 1). \tag{5}$$

We introduce a differential operator

$$\mathcal{S}_n = (\gamma_n I + c_n \sigma_3) \sum_{k=0}^{\infty} {a_n}^{-k-1} \partial_x^k \qquad (n = 1, \ldots, l). \tag{6}$$

By employing the gauge operator $\mathcal{W}$ and the differential operator $\mathcal{S}_n$, we define differential operators $\mathcal{B}_n (n \geqslant 1)$ and $\mathcal{C}_n (n = 1, \ldots, l)$ by

$$\mathcal{B}_n = \left(\mathcal{W} \sigma_3 \partial_x^n \mathcal{W}^{-1}\right)_+ = \sum_{k=0}^{n-1} u_{n-k} \partial_x^k + \sigma_3 \partial_x^n \qquad (n \geqslant 1), \tag{7}$$

$$\mathcal{C}_n = (\mathcal{W} \mathcal{S}_n \mathcal{W}^{-1})_+ = R_n \sum_{k=0}^{\infty} {a_n}^{-k-1} \partial_x^k \qquad (n = 1, \ldots, l), \tag{8}$$

where

$$R_n = \gamma_n I + c_n \left(\sigma_3 + \sum_{l=1}^{\infty} {a_n}^{-l} u_l \right) \qquad (n = 1, \ldots, l). \tag{9}$$

Matrix operators

$$W = I + \sum_{k=1}^{\infty} w_k \lambda^{-k}, \tag{10}$$

$$U = \sigma_3 + \sum_{k=1}^{\infty} u_k \lambda^{-k}, \tag{11}$$

$$S_n = (\gamma_n I + c_n \sigma_3) \sum_{k=0}^{\infty} {a_n}^{-k-1} \lambda^k = -\frac{\gamma_n I + c_n \sigma_3}{\lambda - a_n} \qquad (n = 1, \ldots, l), \tag{12}$$

$$B_n = \sum_{k=0}^{n-1} u_{n-k} \lambda^k + \sigma_3 \lambda^n \qquad (n \geqslant 1), \tag{13}$$

$$C_n = R_n \sum_{k=0}^{\infty} {a_n}^{-k-1} \lambda^k = -\frac{R_n}{\lambda - a_n} \qquad (n = 1, \ldots, l), \tag{14}$$

are obtained from the pseudo-differential operators by replacing ∂_x with λ. We assume that the matrix operators satisfy

$$\partial_{t_n} W = B_n W - W \sigma_3 \lambda^n \qquad (n \geqslant 1), \tag{15}$$

$$\partial_{a_n} W = C_n W - W S_n \qquad (n = 1, \ldots, l), \tag{16}$$

which we call the Sato equation hereafter.

Let us now define a wavefunction.

Definition 2. *A wavefunction $\Psi(\lambda)$ is defined by the following expression:*

$$\Psi(\lambda) = W \Psi_0(\lambda), \tag{17}$$

where

$$\begin{aligned}\Psi_0(\lambda) = {} & \lambda^{\alpha} (\lambda - 1)^{\beta} \prod_{n=1}^{l} (\lambda - a_n)^{\gamma_n} \exp(x\lambda) \\ & \times \operatorname{diag}\left\{ \lambda^{a} (\lambda - 1)^{b} \prod_{n=1}^{l} (\lambda - a_n)^{c_n} \exp\left(\sum_{n=1}^{\infty} t_n \lambda^n\right), \right. \\ & \left. \lambda^{-a} (\lambda - 1)^{-b} \prod_{n=1}^{l} (\lambda - a_n)^{-c_n} \exp\left(-\sum_{n=1}^{\infty} t_n \lambda^n\right)\right\}. \end{aligned} \tag{18}$$

We note that the matrix-valued function $\Psi_0(\lambda)$ satisfies

$$\partial_{a_n} \Psi_0(\lambda) = S_n \Psi_0(\lambda) = \mathcal{S}_n \Psi_0(\lambda) \qquad (n = 1, \ldots, l). \tag{19}$$

This leads to the following theorem:

Proposition 1. *If a matrix operator W satisfies the Sato equation (15) and (16), then the wavefunction $\Psi(\lambda)$ which can be derived from W satisfies the linear systems,*

$$\partial_x \Psi(\lambda) = \lambda \Psi(\lambda), \tag{20}$$

$$\partial_{t_n} \Psi(\lambda) = B_n \Psi(\lambda) \qquad (n \geqslant 1), \tag{21}$$

$$\partial_{a_n} \Psi(\lambda) = C_n \Psi(\lambda) \qquad (n = 1, \ldots, l). \tag{22}$$

The Sato equations also lead to the following theorem:

Proposition 2. *If a matrix operator W satisfies the Sato equation (15) and (16), then the matrix operators U, B_n and C_n satisfy the Lax-type systems,*

$$\partial_{t_n} U = [B_n, U] \qquad (n \geqslant 1), \tag{23}$$

$$\partial_{a_n} U = [C_n, U] \qquad (n = 1, \ldots, l), \tag{24}$$

and the Zakharov–Shabat systems,

$$\partial_{t_m} B_n - \partial_{t_n} B_m + [B_n, B_m] = 0 \qquad (n, m \geqslant 1), \tag{25}$$

$$\partial_{a_m} B_n - \partial_{t_n} C_m + [B_n, C_m] = 0 \qquad (n \geqslant 1, m = 1, \ldots, l), \tag{26}$$

$$\partial_{a_m} C_n - \partial_{a_n} C_m + [C_n, C_m] = 0 \qquad (n, m = 1, \ldots, l). \tag{27}$$

The systems (25) are equal to the Zakharov–Shabat systems in the (1, 1)-reduction of the two-component KP hierarchy. The systems (26) and (27) are the additional ones in the extended hierarchy.

3. The extended two-component system and the sixth Painlevé equation

In this section, we consider a holonomic deformation of systems, obtained from the integrable system given in the previous section. We construct a system of linear differential equations in the spectral parameter λ by using the wavefunction in the extended hierarchy, and then obtain nonlinear systems that describe the condition of the complete integrability of the linear systems. We show that the infinite-dimensional system is reduced to the Schlesinger system, from which P_{VI} is obtained.

If we introduce a differential operator

$$\mathcal{V} = I\left(\alpha - \beta\sum_{k=1}^{\infty}\partial_x^k - \sum_{n=1}^{l}\gamma_n\sum_{k=1}^{\infty}a_n{}^{-k}\partial_x^k + x\partial_x\right) + \sigma_3\left(a - b\sum_{k=1}^{\infty}\partial_x^k - \sum_{n=1}^{l}c_n\sum_{k=1}^{\infty}a_n{}^{-k}\partial_x^k + \sum_{n=1}^{\infty}nt_n\partial_x^n\right), \tag{28}$$

then the matrix-valued function $\Psi_0(\lambda)$ (18) fulfils

$$\lambda\partial_\lambda\Psi_0(\lambda) = \mathcal{V}\Psi_0(\lambda). \tag{29}$$

By using the gauge operator $\mathcal{W}$ and the differential operator $\mathcal{V}$, we define a differential operator $\mathcal{D}$ by

$$\mathcal{D} = (\mathcal{W}\mathcal{V}\mathcal{W}^{-1})_+ = \sum_{k=0}^{\infty} d_k\partial_x^k, \tag{30}$$

where

$$d_0 = \alpha I + a\sigma_3 - b\sum_{l=1}^{\infty}u_l - \sum_{n=1}^{l}c_n\sum_{l=1}^{\infty}a_n{}^{-l}u_l + \sum_{n=1}^{\infty}nt_nu_n,$$

$$d_1 = \left(-\beta - \sum_{n=1}^{l}\gamma_na_n{}^{-1} + x\right)I - b\left(\sigma_3 + \sum_{l=1}^{\infty}u_l\right) - \sum_{n=1}^{l}c_na_n{}^{-1}\left(\sigma_3 + \sum_{l=1}^{\infty}a_n{}^{-l}u_l\right) + t_1\sigma_3 + \sum_{n=2}^{\infty}nt_nu_{n-1},$$

$$d_k = \left(-\beta - \sum_{n=1}^{l} \gamma_n a_n{}^{-k}\right) I - b\left(\sigma_3 + \sum_{l=1}^{\infty} u_l\right)$$
$$- \sum_{n=1}^{l} c_n a_n{}^{-k}\left(\sigma_3 + \sum_{l=1}^{\infty} a_n{}^{-l} u_l\right) + k t_k \sigma_3 + \sum_{n=k+1}^{\infty} n t_n u_{n-k} \qquad (k \geqslant 2). \tag{31}$$

We introduce matrix operators

$$T = \frac{\alpha I + a\sigma_3}{\lambda} + \frac{\beta I + b\sigma_3}{\lambda - 1} + \sum_{n=1}^{l} \frac{\gamma_n I + c_n \sigma_3}{\lambda - a_n} + \sum_{n=1}^{\infty} n t_n \sigma_3 \lambda^{n-1}, \tag{32}$$

$$A = \sum_{k=0}^{\infty} d_k \lambda^{k-1}. \tag{33}$$

We note that

$$\partial_\lambda \Psi_0(\lambda) = T\Psi_0(\lambda). \tag{34}$$

We assume that the matrix operator A satisfies the Sato equation with respect to the spectral parameter:

$$\partial_\lambda W = AW - WT. \tag{35}$$

This leads to the following theorem:

Proposition 3. *If a matrix operator W satisfies the reduction condition (35), then the wavefunction $\Psi(\lambda)$ (17) satisfies the linear system*

$$\partial_\lambda \Psi(\lambda) = A\Psi(\lambda). \tag{36}$$

The Sato equations also lead to the following theorem:

Proposition 4. *If a matrix operator W satisfies the Sato equation* (15), (16) *and* (35), *then the matrix operators U and A satisfy the Lax-type systems,*

$$\partial_\lambda U = [A, U], \tag{37}$$

and the matrix operators A, B_n and C_n satisfy the Zakharov–Shabat-type systems,

$$\partial_{t_n} A - \partial_\lambda B_n + [A, B_n] = 0 \qquad (n \geqslant 1), \tag{38}$$

$$\partial_{a_n} A - \partial_\lambda C_n + [A, C_n] = 0 \qquad (n = 1, \ldots, l). \tag{39}$$

If we introduce matrices

$$P = \alpha I + a\sigma_3 - b\sum_{l=1}^{\infty} u_l - \sum_{n=1}^{l} c_n \sum_{l=1}^{\infty} a_n{}^{-l} u_l + \sum_{n=1}^{\infty} n t_n u_n, \tag{40}$$

$$Q = \beta I + b\left(\sigma_3 + \sum_{l=1}^{\infty} u_l\right), \tag{41}$$

$$T_0 = xI + t_1\sigma_3 + \sum_{n=2}^{\infty} n t_n u_{n-1}, \tag{42}$$

$$T_k = (k+1)t_{k+1}\sigma_3 + \sum_{n=k+2}^{\infty} nt_n u_{n-k-1} \qquad (k \geqslant 1), \tag{43}$$

then we have

$$A = \frac{P}{\lambda} + \frac{Q}{\lambda - 1} + \sum_{n=1}^{l} \frac{R_n}{\lambda - a_n} + \sum_{k=0}^{\infty} T_k \lambda^k, \tag{44}$$

where the matrix R_n is given by (9). If we put $t_n \equiv 0 (n \geqslant r)$, then we have $R_k \equiv 0$ $(k \geqslant r-1)$, and A has a pole of degree r at $\lambda = \infty$. In this case, the linear system (36) is said to have an irregular singular point at $\lambda = \infty$ of Poincaré rank $r - 1$.

By using (14), (39) and (44), we obtain the systems

$$\partial_{a_n} P + \left[\frac{P}{a_n}, R_n\right] = 0, \tag{45}$$

$$\partial_{a_n} Q + \left[\frac{Q}{a_n - 1}, R_n\right] = 0, \tag{46}$$

$$\partial_{a_n} R_m + \left[\frac{R_m}{a_n - a_m}, R_n\right] = 0 \qquad (m \neq n), \tag{47}$$

$$\partial_{a_n} R_n - \left[\frac{P}{a_n} + \frac{Q}{a_n - 1} + \sum_{m=1,\ldots,l, m\neq n} \frac{R_m}{a_n - a_m} + \sum_{l=0}^{\infty} {a_n}^l T_l, R_n\right] = 0, \tag{48}$$

$$\partial_{a_n} T_k - \left[\sum_{l=k+1}^{\infty} {a_n}^{l-k-1} T_l, R_n\right] = 0 \qquad (k \geqslant 0). \tag{49}$$

If we put $t_n \equiv 0$ $(n \geqslant 1)$ and $x \equiv 0$, then the coefficient matrices reduce to $T_k \equiv 0$ $(k \geqslant 0)$ and we have

$$\partial_{a_n} P + \left[\frac{P}{a_n}, R_n\right] = 0, \tag{50}$$

$$\partial_{a_n} Q + \left[\frac{Q}{a_n - 1}, R_n\right] = 0, \tag{51}$$

$$\partial_{a_n} R_m + \left[\frac{R_m}{a_n - a_m}, R_n\right] = 0 \qquad (m \neq n), \tag{52}$$

$$\partial_{a_n} R_n - \left[\frac{P}{a_n} + \frac{Q}{a_n - 1} + \sum_{m=1,\ldots,l, m\neq n} \frac{R_m}{a_n - a_m}, R_n\right] = 0. \tag{53}$$

This system is nothing but the Schlesinger system [16]. If we set $l = 1$, then we have

$$\partial_{a_1} P + \left[\frac{P}{a_1}, R_1\right] = 0, \tag{54}$$

$$\partial_{a_1} Q + \left[\frac{Q}{a_1 - 1}, R_1\right] = 0. \tag{55}$$

This system is equivalent to $\mathrm{P_{VI}}$ in the paper, [4].

4. The two-component KP hierarchy and the other Painlevé equations

In this section, we study holonomic deformation relating to the (1, 1)-reduction of the two-component KP hierarchy. We show that systems obtained from the deformation reduce to the Painlevé equation, P_V, P_{IV}, P_{III} and P_{II}.

4.1. The fifth Painlevé equation

We explain the (1, 1)-reduction of the two-component KP hierarchy. We show that the systems that describe the condition of the holonomic deformation that contains this hierarchy as a part reduces to P_V. Therefore we find that P_V is obtained through the reduction from the nonlinear Schrödinger equation.

We define the gauge operator

$$\mathcal{W} = I + \sum_{k=1}^{\infty} w_k \partial_x^{-k}, \tag{56}$$

whose 2×2 coefficient matrices w_k do not depend on the parameter x. This condition for the coefficients is equivalent to 'the (1, 1)-reduction'. By using the gauge operator $\mathcal{W}$, we define a pseudo-differential operator $\mathcal{U}$ by

$$\mathcal{U} = \mathcal{W}\sigma_3\mathcal{W}^{-1} = \sigma_3 + \sum_{k=1}^{\infty} u_k \partial_x^{-k}. \tag{57}$$

We define a differential operator $\mathcal{B}_n$ by

$$\mathcal{B}_n = \left(\mathcal{W}\sigma_3\partial_x^n\mathcal{W}^{-1}\right)_+ = \sum_{k=0}^{n-1} u_{n-k}\partial_x^k + \sigma_3\partial_x^n \qquad (n \geqslant 1). \tag{58}$$

Matrix operators

$$W = I + \sum_{k=1}^{\infty} w_k \lambda^{-k}, \tag{59}$$

$$U = \sigma_3 + \sum_{k=1}^{\infty} u_k \lambda^{-k}, \tag{60}$$

$$B_n = \sum_{k=0}^{n-1} u_{n-k}\lambda^k + \sigma_3\lambda^n \qquad (n \geqslant 1) \tag{61}$$

are obtained from the pseudo-differential operators by replacing ∂_x with λ. We assume that the matrix operators satisfy the Sato equation

$$\partial_{t_n} W = B_n W - W\sigma_3\lambda^n \qquad (n \geqslant 1). \tag{62}$$

We define a wavefunction

$$\Psi(\lambda) = W\Psi_0(\lambda), \tag{63}$$

where

$$\Psi_0(\lambda) = \lambda^{\alpha}(\lambda-1)^{\beta}\exp(x\lambda)\,\mathrm{diag}\left\{\lambda^a(\lambda-1)^b\exp\left(\sum_{n=1}^{\infty} t_n\lambda^n\right),\right.$$
$$\left.\lambda^{-a}(\lambda-1)^{-b}\exp\left(-\sum_{n=1}^{\infty} t_n\lambda^n\right)\right\}. \tag{64}$$

This definition of the wavefunction is slightly different from the usual one. The difference does not affect the soliton system, but affects the system of the holonomic deformation.

This leads to the following proposition:

Proposition 5. *If a matrix operator W satisfies the Sato equation (62), then the matrix operators U and B_n satisfy*

$$\partial_{t_n} U = [B_n, U] \qquad (n \geqslant 1), \tag{65}$$

$$\partial_{t_m} B_n - \partial_{t_n} B_m + [B_n, B_m] = 0 \qquad (n, m \geqslant 1). \tag{66}$$

Furthermore, the wavefunction $\Psi(\lambda)$ satisfies the linear systems,

$$\partial_x \Psi(\lambda) = \lambda\Psi(\lambda), \tag{67}$$

$$\partial_{t_n} \Psi(\lambda) = B_n \Psi(\lambda) \qquad (n \geqslant 1). \tag{68}$$

We consider the holonomic deformation that contains the two-component system. If we introduce a differential operator

$$\mathcal{V} = I\left(\alpha - \beta \sum_{k=1}^{\infty} \partial_x^k + x\partial_x\right) + \sigma_3\left(a - b\sum_{k=1}^{\infty} \partial_x^k + \sum_{n=1}^{\infty} n t_n \partial_x^n\right), \tag{69}$$

then the matrix-valued function $\Psi_0(\lambda)$ (64) satisfies

$$\lambda\partial_\lambda \Psi_0(\lambda) = \mathcal{V}\Psi_0(\lambda). \tag{70}$$

By using the gauge operator $\mathcal{W}$ and the differential operator $\mathcal{V}$, we define a differential operator $\mathcal{D}$ by

$$\mathcal{D} = (\mathcal{W}\mathcal{V}\mathcal{W}^{-1})_+ = \sum_{k=0}^{\infty} d_k \partial_x^k, \tag{71}$$

where

$$\begin{aligned} d_0 &= \alpha I + a\sigma_3 - b\sum_{l=1}^{\infty} u_l + \sum_{n=1}^{\infty} n t_n u_n, \\ d_1 &= (-\beta + x)I - b\left(\sigma_3 + \sum_{l=1}^{\infty} u_l\right) + t_1\sigma_3 + \sum_{n=2}^{\infty} n t_n u_{n-1}, \\ d_k &= -\beta I - b\left(\sigma_3 + \sum_{l=1}^{\infty} u_l\right) + k t_k \sigma_3 + \sum_{n=k+1}^{\infty} n t_n u_{n-k} \qquad (k \geqslant 2). \end{aligned} \tag{72}$$

We introduce matrix operators

$$T = \frac{\alpha I + a\sigma_3}{\lambda} + \frac{\beta I + b\sigma_3}{\lambda - 1} + \sum_{n=1}^{\infty} n t_n \sigma_3 \lambda^{n-1}, \tag{73}$$

$$A = \sum_{k=0}^{\infty} d_k \lambda^{k-1}. \tag{74}$$

We note that

$$\partial_\lambda \Psi_0(\lambda) = T\Psi_0(\lambda). \tag{75}$$

We assume that the matrix operator A satisfies the condition

$$\partial_\lambda W = AW - WT. \tag{76}$$

This leads to the following proposition:

Proposition 6. *If a matrix operator W satisfies the reduction condition (76), then the matrix operators U, A and B_n satisfy*

$$\partial_\lambda U = [A, U], \tag{77}$$

$$\partial_{t_n} A - \partial_\lambda B_n + [A, B_n] = 0 \qquad (n \geqslant 1). \tag{78}$$

Furthermore, the wavefunction $\Psi(\lambda)$ (63) satisfies the linear system,

$$\partial_\lambda \Psi(\lambda) = A\Psi(\lambda). \tag{79}$$

If we introduce matrices

$$P = \alpha I + a\sigma_3 - b\sum_{l=1}^{\infty} u_l + \sum_{n=1}^{\infty} nt_n u_n, \tag{80}$$

$$Q = \beta I + b\left(\sigma_3 + \sum_{l=1}^{\infty} u_l\right), \tag{81}$$

$$T_0 = xI + t_1\sigma_3 + \sum_{n=2}^{\infty} nt_n u_{n-1}, \tag{82}$$

$$T_k = (k+1)t_{k+1}\sigma_3 + \sum_{n=k+2}^{\infty} nt_n u_{n-k-1} \qquad (k \geqslant 1), \tag{83}$$

then we have

$$A = \frac{P}{\lambda} + \frac{Q}{\lambda - 1} + \sum_{k=0}^{\infty} T_k \lambda^k. \tag{84}$$

By using (61), (78) and (84), we obtain the systems

$$\partial_{t_1} P + [P, u_1] = 0, \tag{85}$$

$$\partial_{t_1} Q + [Q, u_1 + \sigma_3] = 0, \tag{86}$$

$$\partial_{t_1} T_0 - \sigma_3 + [T_0, u_1] + [P + Q, \sigma_3] = 0, \tag{87}$$

$$\partial_{t_1} T_k + [T_k, u_1] + [T_{k-1}, \sigma_3] = 0 \qquad (k \geqslant 1). \tag{88}$$

If we put $t_n \equiv 0$ $(n \geqslant 2)$, then the coefficient matrices reduce to $T_0 = t_1\sigma_3$, $T_k \equiv 0$ $(k \geqslant 1)$, and then we have

$$\partial_{t_1} P + [P, u_1] = 0, \tag{89}$$

$$\partial_{t_1} Q + [Q, u_1 + \sigma_3] = 0. \tag{90}$$

This system is equivalent to $\mathrm{P_V}$ in the paper, [4].

4.2. The fourth Painlevé equation

We consider the different holonomic deformation that relates to the hierarchy in the previous subsection. We show that the system that describes the deformation condition reduces to $\mathrm{P_{IV}}$. This fact follows the result in the paper, [5].

We employ the same soliton system as in the previous subsection. But we define the wavefunction as follows:

$$\Psi(\lambda) = W\Psi_0(\lambda), \tag{91}$$

where

$$\Psi_0(\lambda) = \lambda^\alpha \exp(x\lambda) \begin{pmatrix} \lambda^a \exp\left(\sum_{n=1}^\infty t_n\lambda^n\right) & 0 \\ 0 & \lambda^{-a}\exp\left(-\sum_{n=1}^\infty t_n\lambda^n\right) \end{pmatrix}. \tag{92}$$

This leads to the following proposition:

Proposition 7. *If a matrix operator W satisfies the Sato equation (62), then the wavefunction $\Psi(\lambda)$ satisfies the linear systems,*

$$\partial_x\Psi(\lambda) = \lambda\Psi(\lambda), \tag{93}$$

$$\partial_{t_n}\Psi(\lambda) = B_n\Psi(\lambda) \qquad (n \geqslant 1). \tag{94}$$

We present the reduction condition for the soliton system. If we introduce a differential operator

$$\mathcal{T} = I(\alpha + x\partial_x) + \sigma_3\left(a + \sum_{n=1}^\infty nt_n\partial_x^n\right), \tag{95}$$

then the matrix-valued function $\Psi_0(\lambda)$ (92) satisfies

$$\lambda\partial_\lambda\Psi_0(\lambda) = \mathcal{T}\Psi_0(\lambda). \tag{96}$$

By using the gauge operator $\mathcal{W}$ and the differential operator $\mathcal{T}$, we define a differential operator $\mathcal{A}$ by

$$\mathcal{A} = (\mathcal{W}\mathcal{T}\mathcal{W}^{-1})_+ = \sum_{k=0}^\infty a_k\partial_x^k, \tag{97}$$

where

$$\begin{aligned} a_0 &= \alpha I + a\sigma_3 + \sum_{n=1}^\infty nt_nu_n, \\ a_1 &= xI + t_1\sigma_3 + \sum_{n=2}^\infty nt_nu_{n-1}, \\ a_k &= kt_k\sigma_3 + \sum_{n=k+1}^\infty nt_nu_{n-k} \qquad (k \geqslant 2). \end{aligned} \tag{98}$$

We introduce matrix operators

$$T = I(\alpha + x\lambda) + \sigma_3\left(a + \sum_{n=1}^\infty nt_n\lambda^n\right), \tag{99}$$

$$A = \sum_{k=0}^{\infty} a_k \lambda^k. \tag{100}$$

We assume that the matrix operator A satisfies

$$\lambda \partial_\lambda W = AW - WT. \tag{101}$$

This leads to the following proposition:

Proposition 8. *If a matrix operator W satisfies the reduction condition (101), then the matrix operators U, A and B_n satisfy*

$$\lambda \partial_\lambda U = [A, U], \tag{102}$$

$$\partial_{t_n} A - \lambda \partial_\lambda B_n + [A, B_n] = 0 \qquad (n \geqslant 1). \tag{103}$$

Furthermore, the wavefunction $\Psi(\lambda)$ (91) satisfies the linear system,

$$\lambda \partial_\lambda \Psi(\lambda) = A \Psi(\lambda). \tag{104}$$

Remark 4.1. If we put $t_n \equiv 0$ $(n \geqslant l)$, then we have $a_k \equiv 0$ $(k \geqslant l)$. In this case, the linear system (104) has a regular singular point at $\lambda = 0$ and an irregular singular point at $\lambda = \infty$ of Poincaré rank $l - 1$. Hence we guess that the systems (103) are equivalent to the fourth Painlevé equation with several variables; see [9].

By using (61), (100) and (103), we have the systems

$$\partial_{t_1} a_0 + [a_0, u_1] = 0, \tag{105}$$

$$\partial_{t_1} a_1 - \sigma_3 + [a_1, u_1] + [a_0, \sigma_3] = 0, \tag{106}$$

$$\partial_{t_1} a_k + [a_k, u_1] + [a_{k-1}, \sigma_3] = 0 \qquad (k \geqslant 2). \tag{107}$$

If we put $t_2 \equiv 1/2, t_n \equiv 0$ $(n \geqslant 3)$, then the coefficient matrices reduce to $a_2 = \sigma_3, a_k \equiv 0$ $(k \geqslant 3)$, and we have

$$\partial_{t_1} a_0 + [a_0, u_1] = 0, \tag{108}$$

$$\partial_{t_1} a_1 - \sigma_3 + [a_1, u_1] + [a_0, \sigma_3] = 0. \tag{109}$$

This system is equivalent to P_{IV} in the paper, [4].

4.3. The third Painlevé equation

We present that the system that is the condition of the different holonomic deformation reduces to P_{III}. So we find that P_{III} is obtained through the reduction from the nonlinear Schrödinger equation.

We employ the same soliton system as in the previous subsection, and we give another reduction condition for the soliton system. If we introduce a differential operator

$$\mathcal{T} = I\left(\alpha \partial_x + x \partial_x^2\right) + \sigma_3 \left(a \partial_x + \sum_{n=1}^{\infty} n t_n \partial_x^{n+1}\right), \tag{110}$$

then the matrix-valued function $\Psi_0(\lambda)$ (92) satisfies

$$\lambda^2 \partial_\lambda \Psi_0(\lambda) = \mathcal{T} \Psi_0(\lambda). \tag{111}$$

By using the gauge operator $\mathcal{W}$ and the differential operator $\mathcal{T}$, we define a differential operator $\mathcal{A}$ by

$$\mathcal{A} = (\mathcal{W}\mathcal{T}\mathcal{W}^{-1})_+ = \sum_{k=0}^{\infty} a_k \partial_x^k, \tag{112}$$

where

$$\begin{aligned} a_0 &= -w_1 + a u_1 + \sum_{n=1}^{\infty} n t_n u_{n+1}, \\ a_1 &= \alpha I + a\sigma_3 + \sum_{n=1}^{\infty} n t_n u_n, \\ a_2 &= x I + t_1 \sigma_3 + \sum_{n=2}^{\infty} n t_n u_{n-1}, \\ a_k &= (k-1) t_{k-1} \sigma_3 + \sum_{n=k}^{\infty} n t_n u_{n-k+1} \qquad (k \geqslant 3). \end{aligned} \tag{113}$$

We introduce matrix operators

$$T = I(\alpha\lambda + x\lambda^2) + \sigma_3 \left(a\lambda + \sum_{n=1}^{\infty} n t_n \lambda^{n+1} \right), \tag{114}$$

$$A = \sum_{k=0}^{\infty} a_k \lambda^k. \tag{115}$$

We assume that the matrix operator A satisfies

$$\lambda^2 \partial_\lambda W = AW - WT. \tag{116}$$

This leads to the following proposition:

Proposition 9. *If a matrix operator W satisfies the reduction condition (116), then the matrix operators U, A and B_n satisfy*

$$\lambda^2 \partial_\lambda U = [A, U], \tag{117}$$

$$\partial_{t_n} A - \lambda^2 \partial_\lambda B_n + [A, B_n] = 0 \qquad (n \geqslant 1). \tag{118}$$

Furthermore, the wavefunction $\Psi(\lambda)$(91) satisfies the linear system,

$$\lambda^2 \partial_\lambda \Psi(\lambda) = A\Psi(\lambda). \tag{119}$$

By using (61), (115) and (118), we obtain the systems

$$\partial_{t_1} a_0 + [a_0, u_1] = 0, \tag{120}$$

$$\partial_{t_1} a_1 + [a_1, u_1] + [a_0, \sigma_3] = 0, \tag{121}$$

$$\partial_{t_1} a_2 - \sigma_3 + [a_2, u_1] + [a_1, \sigma_3] = 0, \tag{122}$$

$$\partial_{t_1} a_k + [a_k, u_1] + [a_{k-1}, \sigma_3] = 0 \qquad (k \geqslant 3). \tag{123}$$

If we put $t_n \equiv 0$ $(n \geqslant 2)$, then the coefficient matrices reduce to $a_2 = t_1\sigma_3$, $a_k \equiv 0$ $(k \geqslant 3)$, and then we have

$$\partial_{t_1} a_0 + [a_0, u_1] = 0, \tag{124}$$

$$\partial_{t_1} a_1 + [a_1, u_1] + [a_0, \sigma_3] = 0. \tag{125}$$

We can obtain $\mathrm{P_{III}}$ from this system (4.3).

4.4. The second Painlevé equation

We present that the system that describes the condition of the different holonomic deformation reduces to P_{II}.

We employ the same soliton system as in subsection 4.1. However we define the wavefunction as follows:

$$\Psi(\lambda) = W\Psi_0(\lambda), \tag{126}$$

where

$$\Psi_0(\lambda) = \lambda^\alpha e^{x\lambda}\begin{pmatrix}\exp\left(\sum_{n=1}^\infty t_n\lambda^n\right) & 0\\ 0 & \exp\left(-\sum_{n=1}^\infty t_n\lambda^n\right)\end{pmatrix}. \tag{127}$$

This leads to the following proposition:

Proposition 10. *If a matrix operator W satisfies the Sato equation (62), then the wavefunction $\Psi(\lambda)$ satisfies the linear systems,*

$$\partial_x\Psi(\lambda) = \lambda\Psi(\lambda), \tag{128}$$

$$\partial_{t_n}\Psi(\lambda) = B_n\Psi(\lambda) \qquad (n \geqslant 1). \tag{129}$$

We give the reduction condition for the soliton system. If we introduce a differential operator

$$\mathcal{T} = I\left(\alpha\partial_x^{-1} + x\right) + \sigma_3\sum_{n=1}^\infty nt_n\partial_x^{n-1}, \tag{130}$$

then the matrix-valued function $\Psi_0(\lambda)$ (127) satisfies

$$\partial_\lambda\Psi_0(\lambda) = \mathcal{T}\Psi_0(\lambda). \tag{131}$$

By using the gauge operator $\mathcal{W}$ and the differential operator $\mathcal{T}$, we define a differential operator $\mathcal{A}$ by

$$\mathcal{A} = \left(\mathcal{W}\mathcal{T}\mathcal{W}^{-1}\right)_+ = \sum_{k=0}^\infty a_k\partial_x^k, \tag{132}$$

where

$$\begin{aligned} a_0 &= xI + t_1\sigma_3 + \sum_{n=2}^\infty nt_nu_{n-1},\\ a_k &= (k+1)t_{k+1}\sigma_3 + \sum_{n=k+2}^\infty nt_nu_{n-k-1} \qquad (k \geqslant 1). \end{aligned} \tag{133}$$

We introduce matrix operators

$$T = I(\alpha\lambda^{-1} + x) + \sigma_3\sum_{n=1}^\infty nt_n\lambda^{n-1}, \tag{134}$$

$$A = \sum_{k=0}^\infty a_k\lambda^k. \tag{135}$$

We assume that the matrix operator A satisfies

$$\partial_\lambda W = AW - WT. \tag{136}$$

This leads to the following proposition:

Proposition 11. *If a matrix operator W satisfies the reduction condition (136), then the matrix operators U, A and B_n satisfy*

$$\partial_\lambda U = [A, U], \tag{137}$$

$$\partial_{t_n} A - \partial_\lambda B_n + [A, B_n] = 0 \qquad (n \geqslant 1). \tag{138}$$

Furthermore, the wavefunction $\Psi(\lambda)$ (126) satisfies the linear system,

$$\partial_\lambda \Psi(\lambda) = A\Psi(\lambda). \tag{139}$$

Remark 4.2. If we put $t_n \equiv 0$ $(n \geqslant l)$, then we have $a_k \equiv 0$ $(k \geqslant l-1)$. In this case, the linear system (139) has an irregular singular point at $\lambda = \infty$ of Poincaré rank $l-1$. So we guess that the systems (138) are equivalent to the A_g-system; see [11].

By using (61), (135) and (138), we have the systems

$$\partial_{t_1} a_0 - \sigma_3 + [a_0, u_1] = 0, \tag{140}$$

$$\partial_{t_1} a_k + [a_k, u_1] + [a_{k-1}, \sigma_3] = 0 \qquad (k \geqslant 1). \tag{141}$$

If we put $t_3 \equiv 1/3, t_n \equiv 0$ $(n = 2, n \geqslant 4)$, then the coefficient matrices reduce to $a_2 = \sigma_3, a_k \equiv 0$ $(k \geqslant 3)$, and we have

$$\partial_{t_1} a_0 - \sigma_3 + [a_0, u_1] = 0, \tag{142}$$

$$\partial_{t_1} a_1 + [a_1, u_1] + [a_0, \sigma_3] = 0. \tag{143}$$

This system is equivalent to $\mathrm{P_{II}}$ in the paper, [4].

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